

Analysis of the Effect of Uncertain Average Winds on Cruise Fuel Load

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1 Propagation of Uncertainties

- Introduction
- Problem Statement

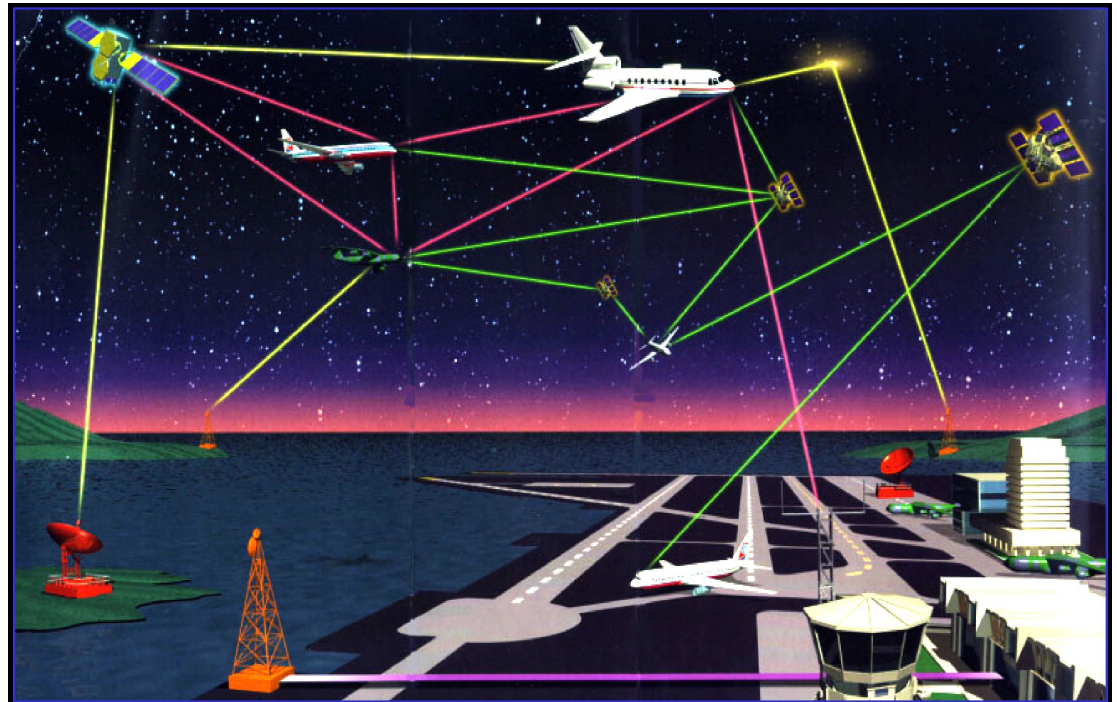
2 Methods and Results

- Generalized Polynomial Chaos (GPC)
- Probability density function Transformation Method (PTM)
- Results

3 Conclusions & Future Work

Introduction

- ATM: a very complex system with a large number of heterogeneous components



Uncertainties in ATM

- ATM operations and in particular aircraft trajectories are subject to many **uncertainties**.
- Sources of uncertainty include:
 - wind and severe weather
 - navigational errors
 - aircraft performance inaccuracies
 - errors in the FMS...

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- ATM operations and in particular aircraft trajectories are subject to many **uncertainties**.
- Sources of uncertainty include:
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 - errors in the FMS...
- The analysis of the impact of uncertainties in aircraft trajectories and its propagation through the flight segments is of great interest
 - Study sensitivity of the system to lack of precise data / measurement errors
 - Aid in the design of a more robust ATM system

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- Analyze promising tools to study uncertainty propagation, comparing with analytical results
- To obtain analytical results, a simplified case has been considered: **Analysis of the Effect of Uncertain Average Winds on Cruise Fuel Load**
 - Wind is the main source of uncertainty in trajectory prediction
 - Mass evolution largely determines fuel consumption and thus flight cost → study uncertainty propagation through mass dynamics
 - Cruise uncertainties have a large impact on the overall flight
- Mass evolution in cruise flight: single **nonlinear equation**, **analytically solvable**. Some interesting conclusions can be derived.

Trajectory uncertainty propagation

- Methods to study trajectory uncertainty propagation fall in two categories
 - parametric: the statistical moments (mean, variance) are propagated
 - non-parametric: the full probability density function is evolved
- In this presentation both type of methods are considered
- Monte-Carlo methods: innumerable works have used these, however they are **not very precise and computationally very intensive**

More methods, more results: take a look at the paper or visit http://complexworld.eu/wiki/Uncertainty_propagation

Mass dynamics in cruise flight

- Assumptions:
 - Cruise flight
 - Other typical flight mechanics assumptions (symmetric flight, parabolic polar, etc)
- Mass equation can be written as $\dot{m} = -(A + Bm^2)$
- Horizontal distance equation:

$$\dot{x} = V + w$$

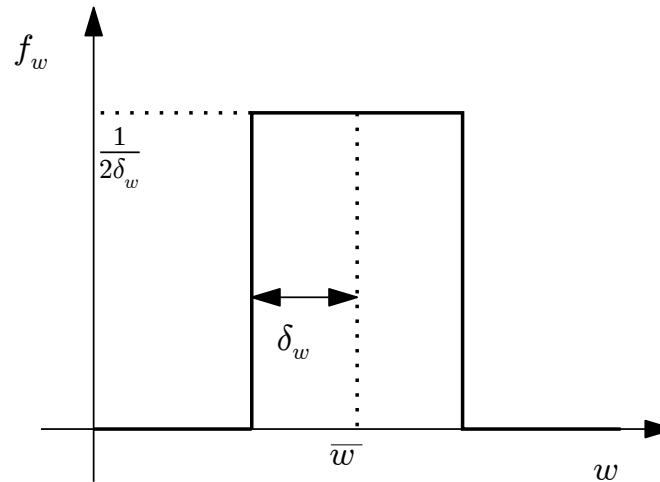
where x is the horizontal distance, and w average wind speed.

- Combining mass and distance equations:

$$\frac{dm}{dx} = -\frac{A + Bm^2}{V + w}$$

- The solution to this equation is used to determine total fuel consumed for a given distance

Uncertainty in average wind

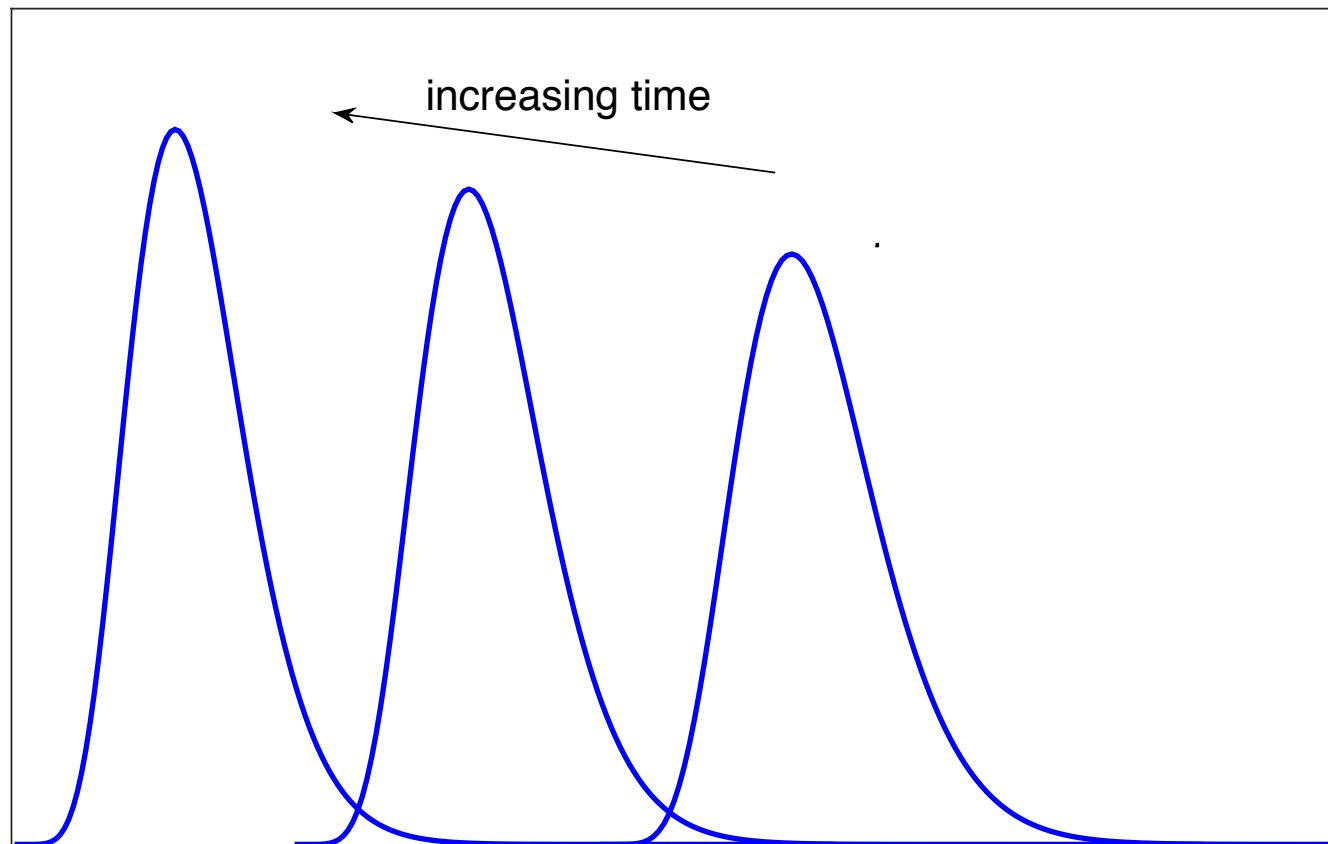


- Consider that the average wind w is not known a priori, but rather a **random variable**
- Objective of this work:

Find how the uncertainty in average wind affects the fuel cost

- Easy **probabilistic model** for w : the Uniform distribution
- $\bar{w}$ is the **mean** and δ_w the width

Propagation of uncertainty



Methods

- The following tools are used to study uncertainty propagation of initial mass uncertainty:
 - Generalized Polynomial Chaos (GPC)
 - A numerical method to propagate distribution functions:
Probability density function Transformation Method (PTM)
- Polynomial chaos is **fast and precise** (compared with Monte Carlo methods). However can only be used to obtain statistical properties such as mean or variance
- **Distribution functions contain all information about the random variable** and thus are very useful to have.

Introduction to GPC I

- Representation of a random process as a **Fourier-type series, with time-dependent coefficients** introduced by Norbert Wiener in 1938
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- What polynomials are best? Choose **orthogonal polynomials with respect to mathematical expectation**. Then the convergence of the series is **exponential**
- Orthogonality: given two polynomials ϕ_i and ϕ_j , then $E[\phi_i\phi_j] = 0$, if $i \neq j$
- For the uniform distribution: **Legendre** polynomials

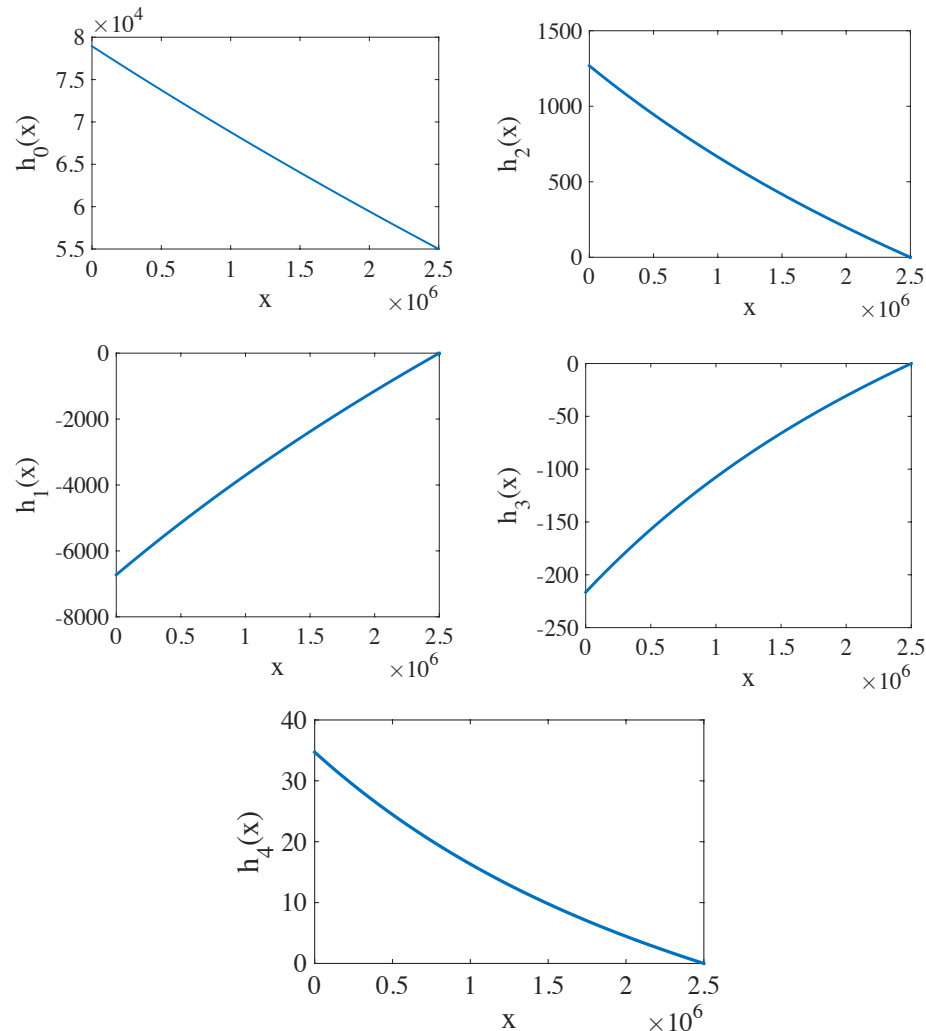
Introduction to GPC II

- Thus expand the mass of the aircraft

$$m(t) = \sum_{i=0}^P h_i(t) L_i$$

- The coefficients $h_i(t)$ verify ODEs which are found from mass differential equation (details in the paper)
- P is the order of the approximation
- Once the coefficients h_i are computed, **calculate mean and variance**
- The advantage of the GPC method is that a small or moderate value of P is enough to get good results, thus resulting in a *computationally much less intensive method than Monte-Carlo simulations*

GPC convergence



- Convergence was obtained with $P = 4$ (see the aircraft parameters in the paper)
- Note how each mode is approximately one order of magnitude smaller than the previous one
- Additional modes do not provide significant changes in the values of the mean or typical deviation

Distribution functions of transformed variables

- The basic idea of PTM is based on the following basic *result* from statistics:

Result

Given a random variable x with probability density function $f_x(x)$, if a new random variable is defined as $y = g(x)$, then $f_y(y)$ is

$$f_y(y) = \frac{f_x(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

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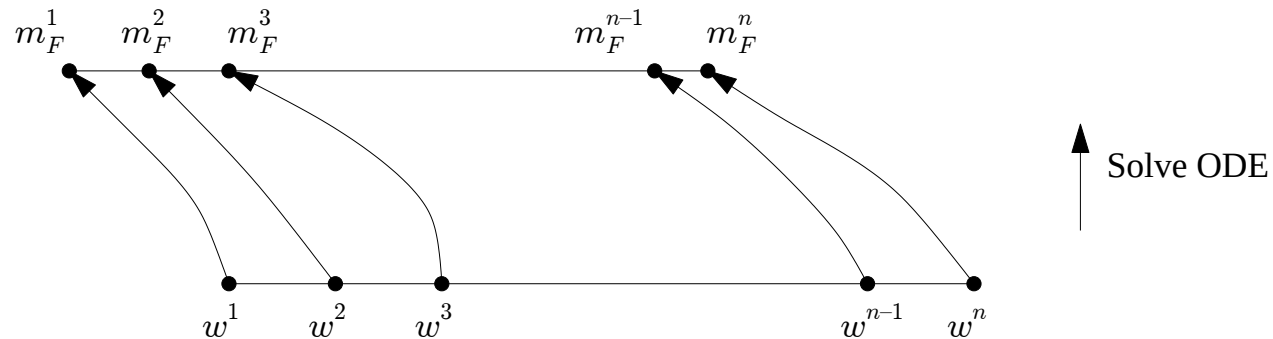
- Denoting $m_F = g(w)$ as the fuel mass obtained by solving the differential equation for each wind w :

$$f_{m_F}(m_F) = \frac{f_w(g^{-1}(m_F))}{|g'(g^{-1}(m_F))|}$$

PTM

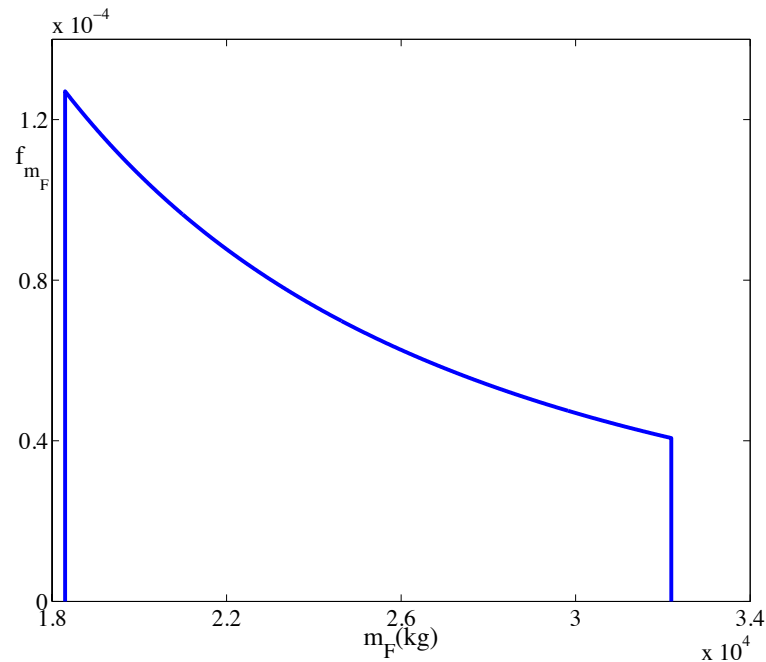
■ Main idea of PTM:

- 1 Take n consecutive points from the domain of the probability density function of w , denoted as w^i
- 2 Compute from the ODE $m_F^i = g(w^i)$



- 3 Simultaneously compute from the ODE the sensitivity function $g'(w^i)$
- 4 Apply the formula, obtaining the value of $f_{m_F}(m_F)$ for the n points.

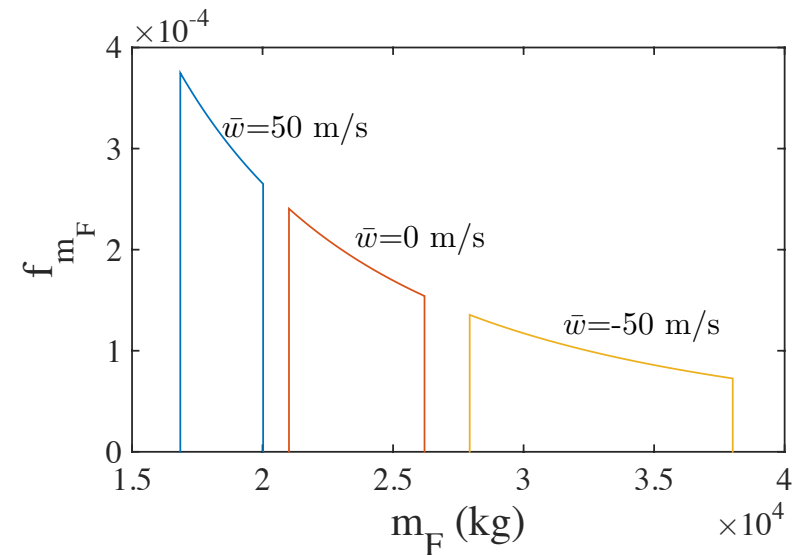
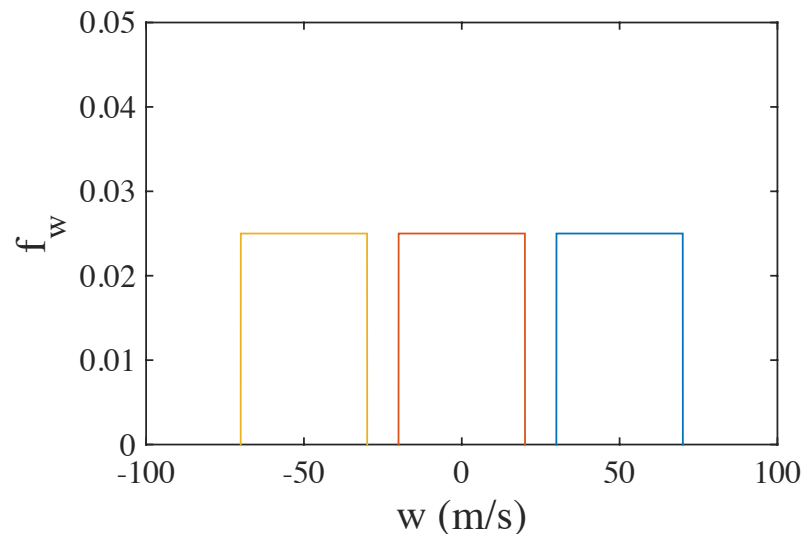
Results



- Numerical parameters used can be found in the paper.
- Analytical results and numerical methods show **excellent agreement**.
- In this example, average wind distribution parameters are $\bar{w} = 0$ m/s, $\delta_w = 50$ m/s.

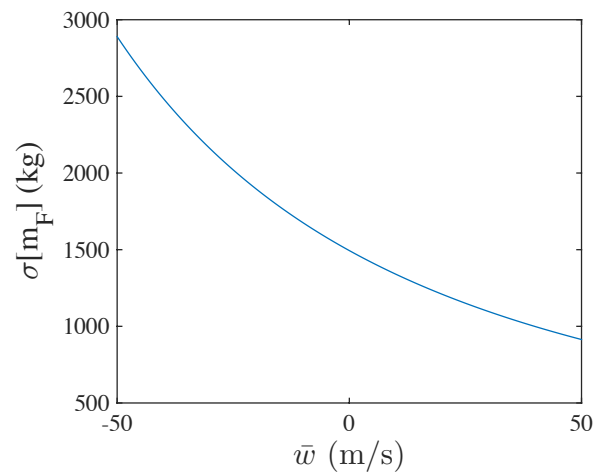
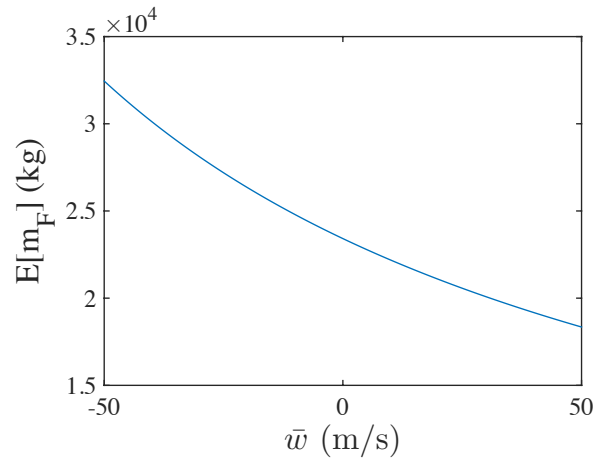
- Resulting statistics: $E[m_F] = 23941.7$ kg and $\sigma[m_F] = 3924.9$ kg.
- Interestingly, the mass of fuel required for the average wind ($w = 0$) is $m_F = 23320.6$ kg $< E[m_F]$. Thus, one concludes **$E[m_F(w)] > m_F(E[w])$** .

Results

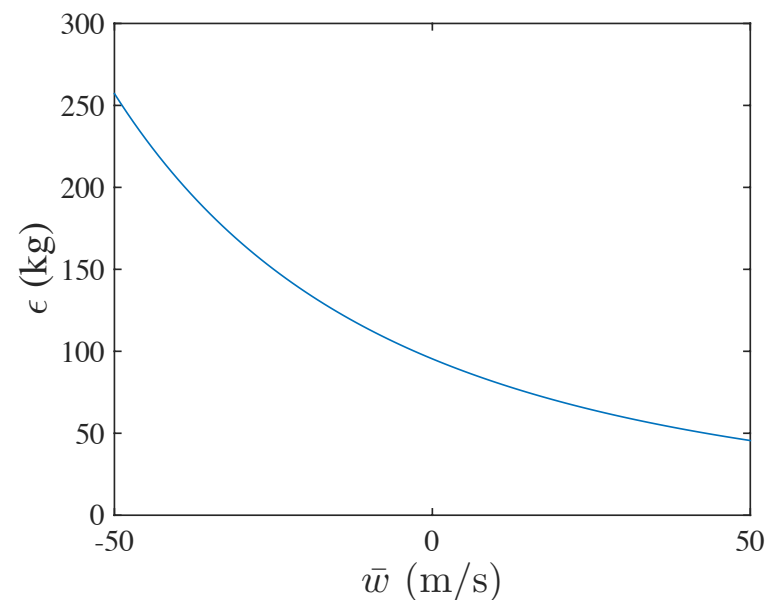


- Other wind distributions with $\bar{w} = -50, 0, 50$ m/s and $\delta_w = 20$ m/s for all cases.
- The shape of the resulting fuel consumption distribution is the same for all.
- In all cases $E[m_F(w)] > m_F(E[w])$.

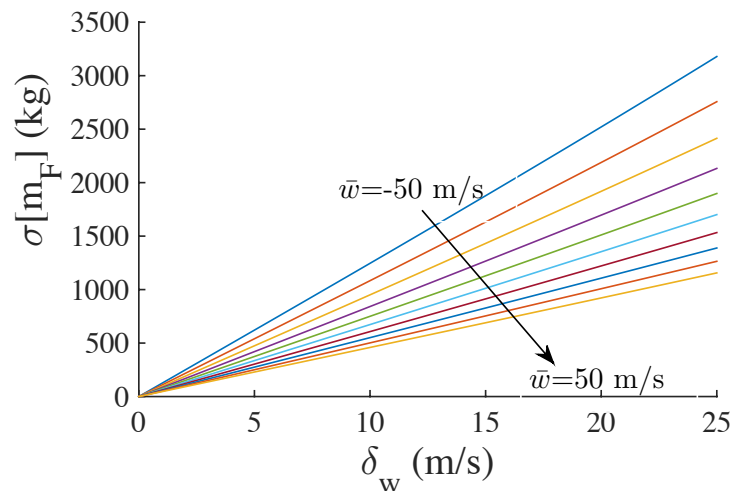
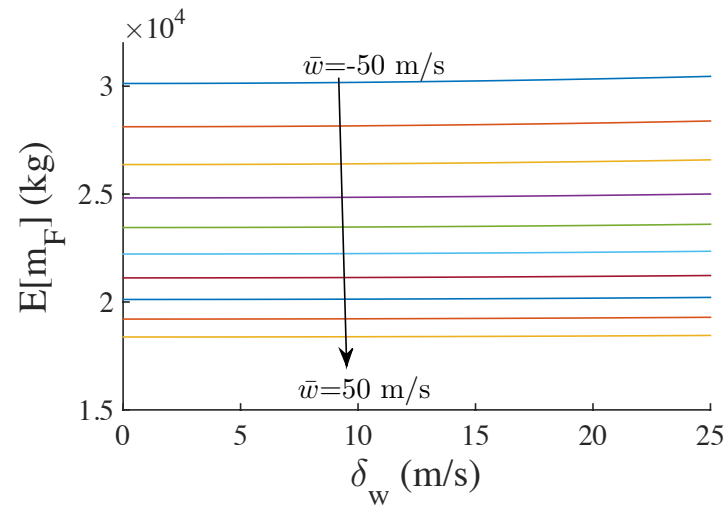
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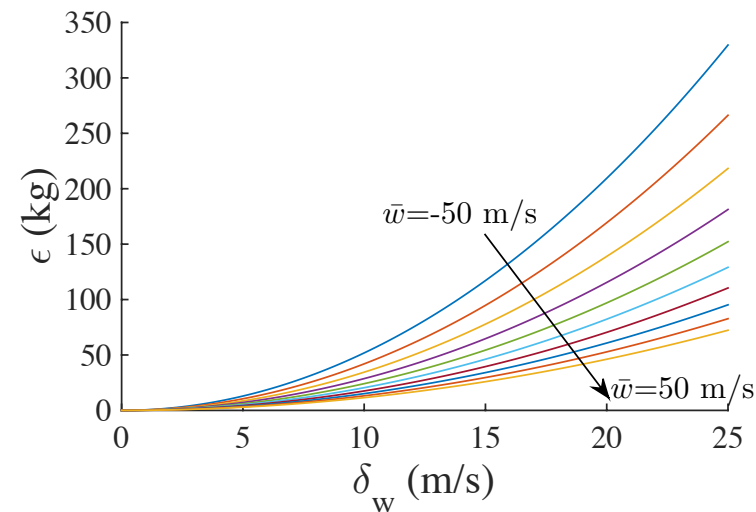
- Fixing $\delta_w = 20$ m/s, effect of $\bar{w}$.
- Uncertainty larger for smaller (negative) $\bar{w}$ so we conclude that head winds increase uncertainty
- In the figure below:
$$\epsilon = E[m_F(w)] - m_F(E[w]) > 0$$



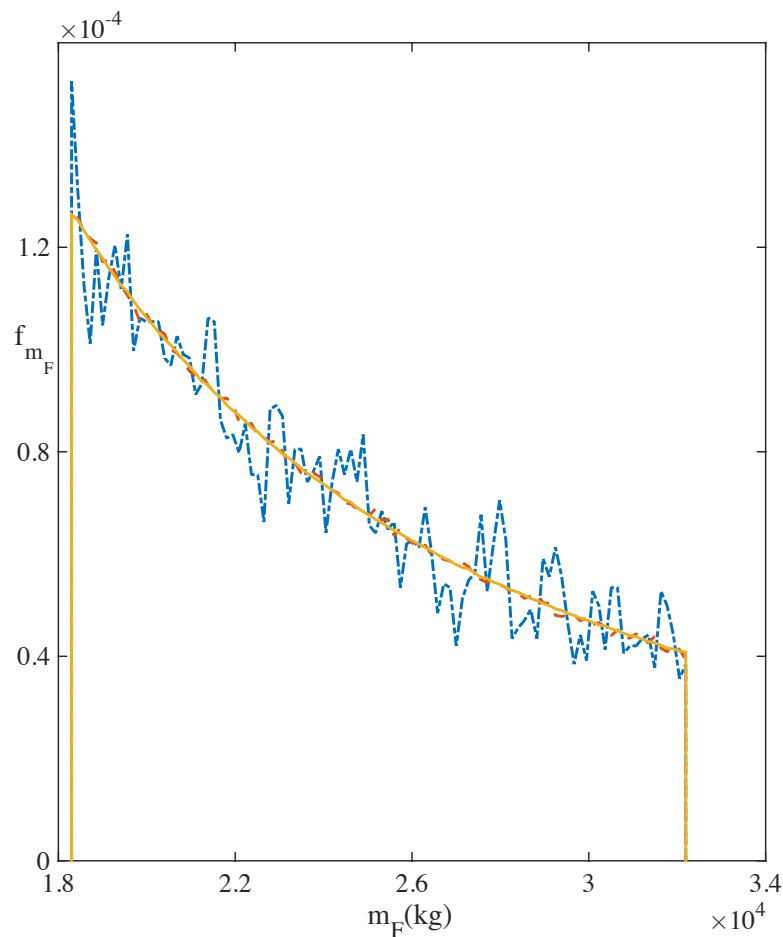
Results



- Simultaneous effect of δ_w and $\bar{w}$
- It can be seen that δ_w has almost no effect on $E[m_F]$ whereas $\sigma[m_F]$ is almost linear with δ_w with slope decreasing as $\bar{w}$ increases.
- ϵ looks quadratic and decreases as $\bar{w}$ increases.



Monte Carlo experiments



- The first example is also analyzed with Monte Carlo simulations, for varying number of experiments N .
- Blue: $N = 10^4$, red: $N = 10^6$, orange: $N = 50 \cdot 10^6$ (coincides with exact result)
- According to Monte Carlo basic estimates $\rightarrow$ errors for each case (with 99% certainty) respectively 100 kg, 10 kg and 1.5 kg.

Computation times

- Comparison of computation time and precision of each method

Method	Computation Time	$E[m_F]$ (kg)	$E[m_F]$ error (%)	$\sigma[m_F]$ (kg)	$\sigma[m_F]$ error (%)
Exact		23941.7		3924.9	
GPC ($P = 4$)	0.15 s	23941.7	10^{-5}	3924.9	10^{-4}
PTM ($n = 1000$)	4 s	23941.7	10^{-10}	3924.9	$2 \cdot 10^{-10}$
Monte Carlo ($N = 10^4$)	31.3 s	23978.6	0.15	3933.4	0.21
Monte Carlo ($N = 10^6$)	≈ 50 min	23938.6	0.013	3917.7	0.18
Monte Carlo ($N = 50 \cdot 10^6$)	≈ 45 h	23941.5	$7 \cdot 10^{-4}$	3925.3	0.011

Conclusions

- Studied the problem of how propagation of average wind uncertainty in cruise flight affects mass consumption, governed by a **nonlinear** ODE
- Evolution of mean and variance obtained using the **GPC method** and distribution functions computed using the **PTM method**

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- Studied the problem of how propagation of average wind uncertainty in cruise flight affects mass consumption, governed by a **nonlinear** ODE
- Evolution of mean and variance obtained using the **GPC method** and distribution functions computed using the **PTM method**
- Results:
 - Analytical and numerical results **show excellent agreement** with **fast computation times**
 - Sometimes Monte Carlo is the only option but in many cases **there are methods much faster than Monte Carlo**
 - The mean of the average fuel consumption is larger than the fuel consumption corresponding to the mean average wind → **quantifies increase of fuel consumption due to uncertainty**
 - Significant difference between head and tail winds → **head winds increase uncertainty, tail winds decrease uncertainty**

Future work

- Consider other flight phases, even simple flight plans
- Consider several additional sources of uncertainty
- Model the effect of feedback (FMS)
- Explore non-intrusive GPC methods (allow to use existing trajectory prediction infrastructure)

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Thanks! Questions?

Visit the wiki:

http://complexworld.eu/wiki/Uncertainty_propagation